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The Fundamental Theorem of Algebra for Monosplines with Multiple Nodes

A. A. ZHENSYKBAEV

Mathematics Department, Kazakh State University, Alma-Ata 480121, USSR

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In this paper we find necessary and sufficient conditions for the existence of monosplines with multiple nodes having zeros at given points with given multiplicities. I. J. Schoenberg [1] proved the fundamental theorem of algebra for monosplines of minimal defect. Later the theorem was extended to other sets of splines (see, for example, [2–4]). C. Micchelli [5] proved this theorem for monosplines with multiple nodes and simple zeros. Then R. B. Barrar and H. L. Loeb [6] extended it to the case where the sum of the maximal multiplicity of zeros and the defect of the monospline is less then its degree. We prove the theorem in the general case.

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Let M^r_{nm} be the set of monosplines of degree r with n nodes of odd multiplicities $m_1, ..., m_n$ $(1 \leqslant m_i \leqslant r)$,

$$M(x) = x^r - \sum_{i=1}^{n} \sum_{j=0}^{m_i - 1} a_{ij}(x - x_i)_+^{r-1-j} + \sum_{k=0}^{r-1} b_k x^k,$$

where $u^s_+ = 0$ if $u \leqslant 0$ and $u^s_+ = u^s$ if $u > 0$, and the coefficients a_{ij}, b_k, and the nodes $0 < x_1 < \cdots < x_n < 1$ are arbitrary.

We define the multiplicity of a zero of the monospline $M \in M^r_{nm}$ in the following way,

$$k^{\pm}(M, x) = \begin{cases} r, & \text{if } M^{(i)}(x \pm 0) = 0 \ (i = 0{:}\ r-1); \\ \min\{i{:}\ M^{(i)}(x \pm 0) \neq 0\}, & \text{otherwise;} \end{cases}$$

$$k_0(M, x) = \max\{k^-(M, x); k^+(M, x)\};$$

$$k(M, 0) = k^+(M, 0), \quad k(M, 1) = k^-(M, 1);$$

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and for $x \in (0, 1)$,

$$k(M, x) = \begin{cases} 2[k_0(M, x)/2] + 1, & \text{if } M \text{ changes sign at the point } x; \\ 2[(k_0(M, x) + 1)/2], & \text{otherwise} \end{cases}$$

($[a]$ is the integral part of a). The point x is called a zero of M if $k(M, x) > 0$. The number $k(M, x)$ is called the multiplicity of the zero x. By $\nu(M)$ we denote the number of zeros of M on $[0, 1]$:

$$\nu(M) = \sum_x k(M, x).$$

For the number $\nu(M)$, $M \in M^r_{nm}$, we have the following estimate (see, for example, [7, p. 331]):

$$\nu(M) \leqslant N + r, \qquad N = \sum_{i=1}^{n} (m_i + 1). \tag{1}$$

Since $(1/r)M' \in M^{r-1}_{nm'}$, $m' = (m'_1, ..., m'_n)$, $m'_i = \min(m_i; r-1)$,

$$\nu(M') \leqslant N + r - 1.$$

If $\nu(M) = N + r$ and $M \in M^r_{nm} \cap C[0, 1]$ then $\nu(M') = N + r - 1$ and the derivative M' has exactly one zero on each interval between neighbouring zeros of M. Thus, we have the next statement.

LEMMA 1. *Let* $M \in M^r_{nm}$, $m_i < r (i = 1 : n)$, $\nu(M) = N + r$, *and let* $0 \leqslant t_1 < \cdots < t_k \leqslant 1$ *be distinct zeros of* M. *Then on each interval* (t_i, t_{i+1}) $(i = 1 : k-1)$ *there is a unique extremal point* τ_i *of* M,

$$M(\tau_i) = \min_{t_i < x < t_{i+1}} M(x) \quad \text{or} \quad M(\tau_i) = \max_{t_i < x < t_{i+1}} M(x),$$

and $M(x)$ *is strongly monotone on* $[0, t_1]$, $[t_k, 1]$, *and on* $[\tau_{i-1}, \tau_i]$ *if* $k(M, t_i)$ *is odd.*

LEMMA 2. *If* $M \in M^r_{nm}$, $\nu(M) = N + r$, *and* M *has* k *zeros* $t_1, ..., t_k$ *of multiplicities* $\rho_1, ..., \rho_k$, $\sum_{i=1}^{k} \rho_i = N + r$, *then there are indices* $k_1 \leqslant \cdots \leqslant k_n$ *such that*

$$\sum_{j=1}^{i} (m_j + 1) - 1 \leqslant \sum_{j=1}^{k_i} \rho_j \leqslant r - m_i + \sum_{j=1}^{i} (m_j + 1) \quad (i = 1 : n). \tag{2}$$

Proof. Let the interval $[0, x_i)$ contain u_i points $t_1, ..., t_{u_i}$ $(0 \leqslant u_i \leqslant k;$

$t_{u_i+1} \geqslant x_i$). The restriction of M on $[0, x_i]$ coincides with some monospline from $M^r_{(i-1)\bar{m}}$, $\bar{m} = (m_1, ..., m_{i-1})$. According to (1)

$$\sum_{j=1}^{u_i} \rho_j + k^-(M, x_i) \leqslant \sum_{j=1}^{i-1} (m_j + 1) + r$$
$$= r - m_i + \sum_{j=1}^{i} (m_j + 1) - 1. \tag{3}$$

On the other hand the interval $(x_i, 1]$ contains the points $t_{v_i}, ..., t_k$, $v_i = u_i + 2 - \operatorname{sgn}(t_{u_i+1} - x_i)$. In view of (1)

$$\sum_{j=v_i}^{k} \rho_j + k^+(M, x_i) \leqslant \sum_{j=i+1}^{n} (m_j + 1) + r.$$

Since $\nu(M) = N + r$

$$\sum_{j=1}^{u_i} \rho_j + \sum_{j=v_i}^{k} \rho_j + k(M, x_i) = N + r$$

and the last inequality implies

$$\sum_{j=1}^{u_i} \rho_j \geqslant \sum_{j=1}^{i} (m_j + 1) - k(M, x_i) + k^+(M, x_i). \tag{4}$$

If $k_0(M, x_i) = k^+(M, x_i) \geqslant 0$ or $k(M, x_i) = 0$ then $0 \leqslant k(M, x_i) - k^+(M, x_i) \leqslant 1$. Define $k_i = u_i$ and inequality (2) follows from (3) and (4).

If $k_0(M, x_i) = k^-(M, x_i) \geqslant 0$ and $k(M, x_i) > 0$ then $0 \leqslant k(M, x_i) - k^-(M, x_i) \leqslant 1$, $v_i = u_i + 2$. Define $k_i = u_i + 1$. In view of (3)

$$\sum_{j=1}^{k_i} \rho_j \leqslant r - m_i + \sum_{j=1}^{i} (m_j + 1) - 1 + k(M, x_i) - k^-(M, x_i)$$
$$\leqslant r - m_i + \sum_{j=1}^{i} (m_j + 1).$$

Inequality (4) implies

$$\sum_{j=1}^{k_i} \rho_j \geqslant \sum_{j=1}^{i} (m_j + 1) + k^+(M, x_i) > \sum_{j=1}^{i} (m_j + 1) - 1.$$

Lemma 2 is proved.

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Now we prove the main result of the paper.

THEOREM. *Let the points* $0 \leqslant t_1 < \cdots < t_k \leqslant 1$ *and the integers* $\rho_1, ..., \rho_k$ $(1 \leqslant \rho_i \leqslant r+1;\ \sum_{i=1}^{k} \rho_i = N+r,\ \ \rho_1 \leqslant r + \operatorname{sgn} t_1,\ \ \rho_k \leqslant r + \operatorname{sgn}(1-t_k))$ *be given. There exists a monospline* $M \in M^r_{nm}$ *with zeros at points* t_i *of multiplicities* ρ_i $(i=1:k)$ *if and only if the interlacing condition* (2) *is satisfied. This monospline* M *is unique in* M^r_{nm}.

Proof. Since Lemma 2 implies the necessity of the interlacing condition, we need only prove the sufficiency. The uniqueness of the monospline M satisfying

$$k(M, t_i) = \rho_i \qquad (i=1:k) \tag{5}$$

was proved by R. B. Barrar and H. L. Loeb [6].

We shall establish the suffciency of (2) by induction on r. The theorem is easily seen to be valid for $r=1$. Assume the theorem is true for the set M^{r-1}_{nm} for all n, m $(m_i \leqslant r-1)$. We shall prove it is valid for M^r_{nm}. By (2) we can choose indices $k_1 \leqslant \cdots \leqslant k_n$ such that

$$\begin{aligned}\sum_{j=1}^{i} (m_j+1) - 1 \leqslant \sum_{j=1}^{k_i} \rho_j \leqslant r - m_i + \sum_{j=1}^{i} (m_j+1)\\ < \sum_{j=1}^{k_i+1} \rho_j \qquad (i=1:n).\end{aligned} \tag{6}$$

The remainder of the proof is broken into two major cases (I and II) based on the way that (2) obtains and within these there are several subcases.

I. Assume

(i) if there exists i: $\sum_{j=1}^{k_i} \rho_j = r - m_i + \sum_{j=1}^{i} (m_j+1)$, then $\rho_{k_i} \leqslant r - m_i$;

(ii) if there exists i: $\sum_{j=1}^{k_i} \rho_j = \sum_{j=1}^{i} (m_i+1) - 1$, then $\rho_{k_i+1} \leqslant r - m_i$.

From (2), (i), and (ii) follows that $m_i < r$ $(i=1:n)$ and $\rho_j \leqslant r$ $(j=1:k)$.

Indeed, if $m_s = r$ $(1 \leqslant s \leqslant n)$ then there is an equality on the right or left side in (2). And according to (i) or (ii) $m_s \leqslant r - \rho_{k_s} \leqslant r-1$ or $m_s \leqslant r - \rho_{k_s+1} \leqslant r-1$. Hence $m_s < r$.

Let us prove that $\rho_j \leqslant r$ $(j=1:k)$. The interlacing conditions (2) imply that

$$\sum_{j=1}^{k_1} \rho_j \leqslant r+1, \qquad \sum_{j=k_n+1}^{k} \rho_j \leqslant r+1, \tag{7}$$

$$\sum_{j=k_i+1}^{k_{i+1}} \rho_j \leqslant r+2 \qquad (i=1:n-1). \tag{8}$$

Let $s \leqslant k_1$ (or $s \geqslant k_n+1$). If there is a strong inequality in (7) then $\rho_s \leqslant r$. If the first (or second) sum in (7) equals $r+1$ then in view of (i) (or (ii)) $1 \leqslant \rho_{k_1} < r$ $(1 \leqslant \rho_{k_n+1} < r)$, hence $\rho_s \leqslant r$.

Let $k_i < s \leqslant k_{i+1}$ $(1 \leqslant i \leqslant n-1)$. If there is an equality in (8) then

$$\sum_{j=1}^{k_{i+1}} \rho_j = r - m_{i+1} + \sum_{j=1}^{i+1} (m_j+1), \quad \sum_{j=1}^{k_i} \rho_j = \sum_{j=1}^{i} (m_j+1) - 1. \tag{9}$$

According to (i) and (ii) $\rho_{k_{i+1}} < r$, $\rho_{k_i+1} < r$, and hence $\rho_s \leqslant r$. If the sum in (8) equals $r+1$ then one of two equalities in (9) is true and in view of (i) or (ii) $\rho_{k_{i+1}} < r$ or $\rho_{k_i+1} < r$. Hence $\rho_s \leqslant r$.

1. Let $m_i < r-1$ $(i=1:n)$ and $\rho_j < r$ $(j=2:k-1)$. We construct the continuous map φ: $T \to T$, $T=[t_1, t_2] \times \cdots \times [t_{k-1}, t_k]$. in the following way. For $\tau \in T$, $\tau = (\tau_1, ..., \tau_{k-1})$, $\tau_i \in [t_i, t_{i+1}]$, we extract the different points $0 \leqslant t'_1 < \cdots < t'_{k'} \leqslant 1$ of the set $\{t_1, ..., t_k, \tau_1, ..., \tau_{k-1}\}$. Let

$$\rho'_1 = \begin{cases} r - m_1 - 1 & \text{if} \quad \rho_1 = r, \tau_1 = 0 \\ \rho_1 - \operatorname{sgn}(\tau_1 - t_1) & \text{otherwise} \end{cases}$$

$$\rho'_{k'} = \begin{cases} r - m_n - 1 & \text{if} \quad \rho_k = r, \tau_{k-1} = 1 \\ \rho_k - \operatorname{sgn}(t_k - \tau_{k-1}) & \text{otherwise} \end{cases}$$

$$\rho'_i = \begin{cases} 1 \quad \text{if} \quad t'_i = \tau_{j_i}, t_{j_i} < \tau_{j_i} < t_{j_i+1} \ (1 < i < k') \\ \rho_{j_i} + 1 - \operatorname{sgn}(t_{j_i} - \tau_{j_i-1}) - \operatorname{sgn}(\tau_{j_i} - t_{j_i}) \qquad \text{if} \quad t'_i = t_{j_i}. \end{cases}$$

$n' = n - (1 - \operatorname{sgn} \tau_1)(1 - \operatorname{sgn}(r - \rho_1)) - (1 - \operatorname{sgn}(1 - \tau_{k-1}))(1 - \operatorname{sgn}(r - \rho_k))$, $m'_i = m_{i+1}$ if $\rho_1 = r$, and $\tau_1 = 0$ and $m'_i = m_i$ otherwise $(i = 1 : n')$.

The integers $\rho'_i \leqslant r$ satisfy the interlacing conditions (2) for $M^{r-1}_{n'm'}$ and

$$\sum_{i=1}^{k'} \rho_i = \sum_{i=1}^{n'} (m'_i + 1) + (r-1) =: N' + (r-1). \tag{10}$$

Indeed, let $\rho_1 < r$ or $\tau_1 > 0$ such that $\rho'_1 = r - \operatorname{sgn}(\tau_1 - t_1)$ $m'_i = m_i$ $(i = 1 : n')$. Thus for s: $t'_s = t_{j_s} < t_k$ we have

$$\sum_{j=1}^{s} \rho'_j = \sum_{j=1}^{j_s} \rho_j - \operatorname{sgn}(\tau_1 - t_1) + \sum_{j=2}^{j_s} (1 - \operatorname{sgn}(\tau_j - t_j)$$
$$- \operatorname{sgn}(t_j - \tau_{j-1})) + \sum_{j=1}^{j_s} \operatorname{sgn}(\tau_j - t_j) \operatorname{sgn}(t_{j+1} - \tau_j)$$
$$= \sum_{j=1}^{j_s} \rho_j - \operatorname{sgn}(\tau_{j_s} - t_{j_s}) + \sum_{j=1}^{j_s-1} (1 - \operatorname{sgn}(\tau_j - t_j))$$
$$\cdot (1 - \operatorname{sgn}(t_{j+1} - \tau_j)),$$

and

$$\sum_{j=1}^{s} \rho'_j = \sum_{j=1}^{j_s} \rho_j - \operatorname{sgn}(\tau_{j_s} - t_{j_s}). \tag{11}$$

Using (11) we obtain

$$\sum_{j=1}^{k'} \rho'_j = \sum_{j=1}^{k-1} \rho_j - \operatorname{sgn}(\tau_{k-1} - t_{k-1}) + \operatorname{sgn}(\tau_{k-1} - t_{k-1})$$
$$\cdot \operatorname{sgn}(t_k - \tau_{k-1}) + \rho_k - \operatorname{sgn}(t_k - \tau_{k-1})$$
$$- (m_n + 1)(1 - \operatorname{sgn}(1 - \tau_{k-1}))(1 - \operatorname{sgn}(r - \rho_k))$$
$$= \sum_{j=1}^{k} (\rho_j - \operatorname{sgn}(\tau_{k-1} - t_{k-1})(1 - \operatorname{sgn}(t_k - \tau_{k-1}))$$
$$- \operatorname{sgn}(t_k - \tau_{k-1}) - (m_n + 1)$$
$$\cdot (1 - \operatorname{sgn}(1 - \tau_{k-1}))(1 - \operatorname{sgn}(r - \rho_k))$$
$$= N + r - 1 - (m_n + 1)(1 - \operatorname{sgn}(1 - \tau_{k-1}))(1 - \operatorname{sgn}(r - \rho_k))$$
$$= N' + (r - 1).$$

Now, letting $t_{k_i} = t'_{u_i}$ we define

(a) $k'_i = u_i - 1$ if $t_{k_i} = \tau_{k_i}$ and $\sum_{j=1}^{k_i} \rho_j = r - m_i + \sum_{j=1}^{i} (m_j + 1)$,

(b) $k'_i = u_i + 1$ if $t_{k_i} < \tau_{k_i}$ and $\sum_{j=1}^{k_i} \rho_j = \sum_{j=1}^{i} (m_j + 1) - 1$,

(c) $k'_i = u_i$ otherwise.

In case (a) using (11) in view of (i) we find

$$\sum_{j=1}^{k'_i} \rho'_j = \sum_{j=1}^{k_i} \rho_j - \operatorname{sgn}(\tau_{k_i} - t_{k_i}) - \rho'_{u_i}$$
$$= r - m_i + \sum_{j=1}^{i} (m_j + 1) - \rho'_{u_i}$$

$$= r - m_i' + \sum_{j=1}^{i} (m_j'+1) - \operatorname{sgn}(\tau_{k_i-1} - t_{k_i-1})$$

$$\cdot \operatorname{sgn}(t_{k_i} - \tau_{k_i-1}) - \rho_{k_i} - 1 + \operatorname{sgn}(\tau_{k_i} - t_{k_i}) + \operatorname{sgn}(t_{k_i} - \tau_{k_i-1})$$

$$= (r-1) - m_i' + \sum_{j=1}^{i} (m_j'+1) + \operatorname{sgn}(t_{k_i} - \tau_{k_i-1})$$

$$\cdot (1 - \operatorname{sgn}(\tau_{k_i-1} - t_{k_i-1})) - \rho_{k_i}$$

$$\leqslant (r-1) - m_i' + \sum_{j=1}^{i} (m_j'+1),$$

$$\sum_{j=1}^{k_i'} \rho_j' \geqslant (r-1) + m_i' + \sum_{j=1}^{i} (m_j'+1) - \rho_{k_i}$$

$$\geqslant (r-1) - m_i' + \sum_{j=1}^{i} (m_j'+1) - (r - m_i)$$

$$= \sum_{j=1}^{i} (m_j'+1) - 1.$$

In a similar way we can verify equality (10) and the inequality

$$\sum_{j=1}^{i} (m_j'+1) - 1 \leqslant \sum_{j=1}^{k_i'} \rho_j' \leqslant (r-1) - m_i' + \sum_{j=1}^{i} (m_j'+1) \quad (i = 1 : n')$$

in cases (b), (c), and for $\rho_1 = r$, $\tau_1 = 0$ (for $\rho_1 = r$ and $\tau_1 = 0$ we define $k_i' = u_{i+1} - 1$ or $k_i' = u_{i+1} + 1$ or $k_i' = u_{i+1}$ corresponding to (a) or (b) or (c)).

Therefore by the inductive hypothesis for every $\tau \in T$ there exists a unique monospline $M_\tau \in M_{n'm'}^{r-1}$ with zeros t_i' of multiplicities ρ_i':

$$k(M_\tau, t_i') = \rho_i' \qquad (i\colon \rho_i' > 0,\ 1 \leqslant i \leqslant k').$$

On each interval $[t_i, t_{i+1}]$ we distinguish the points ξ_i, η_i in the following way:

$$\xi_i = t_i \text{ if } t_i = \tau_i; \qquad \eta_i = t_{i+1} \text{ if } \tau_i = t_{i+1}.$$

If $t_i < \tau_i$ then ξ_i is the extremal point of M_τ on $[t_i, \tau_i)$ and $M_\tau(\xi_i) \neq 0$; if $\tau_i < t_{i+1}$ then η_i is the extremal point of M_τ on $(\tau_i, t_{i+1}]$ and $M_\tau(\eta_i) \neq 0$. By Lemma 1 the points ξ_i, η_i are defined uniquely. The monospline M_τ is strongly monotone on $[\xi_i, \eta_i]$ because $k(M_\tau, \tau_i) = 1$ if $t_i < \tau_i < t_{i+1}$. M_τ is continuous on $[t_i, t_{i+1}]$. Hence there is a unique point $\varphi_i \in (\xi_i, \eta_i)$ such that

$$\int_{t_i}^{t_{i+1}} M_\tau(t)\,dt = M_\tau(\varphi_i)(t_{i+1} - t_i) \quad (i = 1 : k-1).$$

Now for every point $\tau \in T$ we assign the point $\varphi(\tau) \in T$, $\varphi(\tau) = (\varphi_1, ..., \varphi_{k-1})$. Below (Lemma 3) we prove the continuity of the map φ: $T \to T$. By Brouwer's theorem there exists a point $\tau^* \in T$ such that $\varphi(\tau^*) = \tau^* : \varphi_i = \tau_i^*$ $(i = 1 : k-1)$. Thus, $\tau_i^* \in (t_i, t_{i+1})$, $M_{\tau^*} \in M_{nm}^{r-1}$, and

$$\int_{t_i}^{t_{i+1}} M_{\tau^*}(t)\,dt = M_{\tau^*}(\tau_i^*)(t_{i+1} - t_i) = 0 \quad (i = 1 : k-1).$$

The monospline

$$M(x) = \int_{t_1}^{x} M_{\tau^*}(t)\,dt$$

satisfies the theorem.

LEMMA 3. *The map $\varphi(\tau)$ is continuous on T.*

Proof. At first we remark that if $M \in M_{nm}^r$ has $N + r$ zeros, $\nu(M) = N + r$, then the the coefficients a_{ij}, b_k in the representation of M are bounded,

$$|a_{ij}| < K, \qquad |b_k| < K \qquad (i = 1 : n; j = 0 : m_i - 1; k = 0 : r-1),$$

where K depends only on r. This simple fact is shown for example, as it was proved in [7, p. 341] in the case $k(M, x) \leqslant I$, i.e., when M has simple zeros. Thus, from an arbitrary sequence $\{M_k\}$, $M_k \in M_{nm}^r$, $\nu(M_k) = N + r$, we can extract a convergent subsequence $\{M_{k_m}\}$ in the following sense: there exist a finite set of points $u_1, ..., u_s$ on $[0, 1]$ such that $\{M_{k_m}\}$ converges uniformly on the arbitrary compact set $G \subset [0, 1] \backslash \{u_1, ..., u_s\}$.

Let $\tau^{(s)} \to \tau^{(0)}$, $\tau^{(s)} \in T$, and let $k'(s)$, $t_i'(s)$, $\rho_i'(s)$, $n'(s)$, $m_i'(s)$, $N'(s)$, $M_s(x) \in M_{n'(s)m'(s)}^{r-1}$ be the quantities corresponding to $\tau^{(s)}$. Since $\nu(M_s) = N'(s) + (r-1)$, from the sequence $\{M_s\}$ we can extract a convergent subsequence $\{M_{s_l}\} \to \bar{M}$, $\bar{M} \in M_{\bar{n}\bar{m}}^{r-1}$, $\bar{m} = (\bar{m}_1, ..., \bar{m}_{\bar{n}})$. If $k(M_s, 0) = r-1$ then $M_s(t) = t^{r-1}$, $t \in (0, x_1(s))$, where $x_1(s)$ is the first knot of M_s and $0 < x_1(s) \leqslant \tau_1^{(s)}$ if $\tau_1^{(s)} > 0$. Hence $x_1(s) \to 0$ if $\tau_1^{(s)} \to 0$. Similarly $x_{n'(s)}(s) \to 1$ if $\tau_{k-1}^{(s)} \to 1$ and $k(M_s, 1) = r-1$. Thus, $\bar{n} \leqslant n'(0)$,

$$\bar{N} := \sum_{i=1}^{\bar{n}} (\bar{m}_i + 1) \leqslant N'(0),$$

and $M_{\bar{n}\bar{m}}^{r-1} \subset M_{n(0)\,m(0)}^{r-1}$.

Further, $t_i'(s) \to t_i'(0)$, $k(\bar{M}, t_i'(0)) \geqslant \rho_i'(0)$, and

$$\bar{N} + (r-1) \geqslant v(\bar{M}) \geqslant \sum_{i=1}^{k'(0)} \rho_i'(0) = N'(0) + (r-1) \geqslant \bar{N} + (r-1).$$

Hence, $k(\bar{M}, t_i'(0)) = \rho_i'(0)$.

On the other hand, $M_0 \in M^{r-1}_{n(0)m(0)}$, $k(M_0, t_i'(0)) = \rho_i'(0)$. In view of the uniqueness [6] $\bar{M}(x) = M_0(x)$, therefore $\varphi(\tau^{(s)}) \to \varphi(\tau^{(0)})$. Lemma 3 is proved.

2. Let $m_s = r-1$ and $m_i < r-1$ $(i \neq s)$, $\rho_j < r$ $(j = 2 : k-1)$. From (2) it follows that

$$\sum_{j=1}^{s} (m_j + 1) - 1 \leqslant \sum_{j=1}^{k_s} \rho_j \leqslant 1 + \sum_{j=1}^{s} (m_j + 1).$$

(a) Let

$$\sum_{j=1}^{k_s} \rho_j = \sum_{j=1}^{s} (m_j + 1) - 1.$$

By (ii) we have $\rho_{k_s+1} = 1$ and

$$\sum_{j=1}^{k_s+1} \rho_j = r + \sum_{j=1}^{s-1} (m_j + 1) =: r + N',$$

$$\sum_{j=k_s+2}^{k} \rho_j = r + \sum_{j=s+1}^{n} (m_j + 1) =: r + N''.$$

The integers $\rho_1, ..., \rho_{k_s+1}, k_1, ..., k_{s-1}, m_1, ..., m_{s-1}$ satisfy the interlacing conditions (2) for $i = 1 : s-1$ and the integers $\rho_{k_s+2}, ..., \rho_k, k_{s+1}, ..., k_n, m_{s+1}, ..., m_n$ satisfy the corresponding interlacing conditions for $M^r_{n-sm''}$ where $m'' = (m_{s+1}, ..., m_n)$. Therefore by part I there exist monosplines $M_1 \in M^r_{s-1m'}$, $m' = (m_1, ..., m_{s-1})$, and $M_2 \in M^r_{n-sm''}$ satisfying (5) for $i = 1 : k_s + 1$ and for $i = k_s + 2 : k$, respectively.

Since $r = m_s + 1$ the number r is even. By Lemma 1 M_1 and M_2 are strongly monotone on (t_{k_s+1}, t_{k_s+2}) and $M_1(x) \cdot M_2(x) > 0$, $x \in (t_{k_s+1}, t_{k_s+2})$. Therefore there exists a unique point x_s in this interval such that $M_1(x_s) = M_2(x_s)$. The monospline

$$M(x) = \begin{cases} M_1(x), & x \in [0, x_s], \\ M_2(x), & x \in (x_s, 1] \end{cases} \tag{12}$$

satisfies the requirements (5) and $M \in M^r_{nm}$.

(b) Let

$$\sum_{j=1}^{k_s} \rho_j = \sum_{j=1}^{s} (m_j + 1).$$

Then

$$\sum_{j=1}^{k_s} \rho_j = r + \sum_{j=1}^{s-1} (m_j + 1), \qquad \sum_{j=k_s+1}^{k} \rho_j = r + \sum_{j=s+1}^{n} (m_j + 1),$$

and we can construct monosplines $M_1 \in M^r_{s-1m'}$ and $M_2 \in M^r_{n-sm''}$ satisfying (5) for $i = 1 : k_s$ and for $i = k_s + 1 : k$, respectively. There exists a point $x_s \in (t_{k_s}, t_{k_s+1})$ for which $M_1(x_s) = M_2(x_s)$. The monospline (12) satisfies the requirements of the theorem.

(c) Let

$$\sum_{j=1}^{k_s} \rho_j = 1 + \sum_{j=1}^{s} (m_j + 1).$$

By (i) $\rho_{k_s} = 1$ and

$$\sum_{j=1}^{k_s-1} \rho_j = r + \sum_{j=1}^{s-1} (m_j + 1), \qquad \sum_{j=k_s}^{k} \rho_j = r + \sum_{j=s+1}^{n} (m_j + 1).$$

We construct monosplines $M_1 \in M^r_{s-1m'}$ and $M_2 \in M^r_{n-sm''}$ and then $M(x) \in M^r_{nm}$ for which $k(M, t_i) = \rho_i$ $(i = 1 : k)$.

3. Let $\rho_s = r$ $(1 < s < k)$, $\rho_j < r$ $(1 < j < r;\ i \neq s)$. There exists an index i such that $k_{i-1} < s \leqslant k_i$. In view of (6) we have

$$\begin{aligned} \sum_{j=1}^{i-1} (m_j + 1) - 1 \leqslant \sum_{j=1}^{k_{i-1}} \rho_j \leqslant \sum_{j=1}^{s-1} \rho_j = \sum_{j=1}^{s} \rho_j - r \\ \leqslant \sum_{j=1}^{k_i} \rho_j - r \leqslant 1 + \sum_{j=1}^{i-1} (m_j + 1). \end{aligned}$$

If

$$\sum_{j=1}^{s-1} \rho_j = \sum_{j=1}^{i-1} (m_j + 1) - 1,$$

then $k_{i-1} = s - 1$ and by (ii) $\rho_{k_{i-1}+1} \leqslant r - m_{i-1} \leqslant r - 1$. But $\rho_{k_{i-1}+1} = \rho_s = r$. If

$$\sum_{j=1}^{s-1} \rho_j = \sum_{j=1}^{i-1} (m_j + 1) + 1,$$

then $s=k_i$ and by (i) $\rho_{k_i} \leqslant r-m_i \leqslant r-1$. But $\rho_{k_i}=\rho_s=r$. Thus

$$\sum_{j=1}^{s-1} \rho_j=\sum_{j=1}^{i-1} (m_j+1).$$

If $s-1>k_{i-1}$ then by (6)

$$r-m_{i-1}+\sum_{j=1}^{i-1} (m_j+1)<\sum_{j=1}^{s-1} \rho_j=\sum_{j=1}^{i-1} (m_j+1).$$

This contradiction shows $s-1=k_{i-1}$. Thus

$$\sum_{j=1}^{s} \rho_j=r+\sum_{j=1}^{i-1} (m_j+1), \quad \sum_{j=s}^{k} \rho_j=r+\sum_{j=i}^{n} (m_j+1).$$

By parts 1 and 2 we can construct the monosplines $M_1 \in M^r_{i-1m'}$ and $M_2 \in M^r_{n-i+1m''}$ satisfying (5) for $i=1:s$ and for $i=s:k$, respectively. The monospline (12), where $x_s=t_s$, satisfies the requirements of the theorem.

II. 1. Let $\rho_{k_s}>r-m_s$,

$$\sum_{j=1}^{k_s} \rho_j=r-m_s+\sum_{j=1}^{s} (m_j+1),$$

and assume (i), (ii) hold for $i \neq s$. Then

$$\sum_{j=1}^{k_s-1} \rho_j+(\rho_{k_s}-1)=r+\sum_{j=1}^{s-1} (m_j+1),$$

$$\sum_{j=k_s+1}^{k} \rho_j+(r-m_s)=r+\sum_{j=s+1}^{n} (m_j+1).$$

In accordance with Part I, we construct two monosplines $M_1 \in M^r_{s-1m'}$ and $M_2 \in M^r_{n-sm''}$, $m'=(m_1, ..., m_{s-1})$, $m''=(m_{s+1}, ..., m_n)$ such that

$$k(M_1, t_i)=\rho_i\ (i=1:k_s-1), \quad k(M_1, t_{k_s})=\rho_{k_s}-1,$$

$$k(M_2, t_i)=\rho_i\ (i=k_s+1:k), k(M_2, t_{k_s})=r-m_s.$$

The monospline (12), where $x_s=t_{k_s}$, satisfies the requirements of the theorem.

2. Let $\rho_{k_{s+1}}>r-m_s$,

$$\sum_{j=1}^{k_s} \rho_j=\sum_{j=1}^{s} (m_j+1)-1,$$

and assume (i), (ii) hold for $i \neq s$. Then

$$\sum_{j=1}^{k_s} \rho_j + (r - m_s) = r + \sum_{j=1}^{s-1} (m_j + 1),$$

$$\sum_{j=k_s+1}^{k} \rho_j - 1 = r + \sum_{j=s+1}^{n} (m_j + 1)$$

and we can construct monosplines $M_1 \in M^r_{s-1m'}$ and $M_2 \in M^r_{n-sm''}$ such that

$$k(M_1, t_i) = \rho_i \ (i = 1 : k_s), \qquad k(M_1, t_{k_s+1}) = r - m_s,$$

$$k(M_2, t_i) = \rho_i \ (i = k_s + 2 : k), \quad k(M_2, t_{k_s+1}) = \rho_{k_s+1} - 1.$$

The monospline (12), where $x_s = t_{k_s+1}$, satisfies (5).

3. Let $s_1 < \cdots < s_u$ be indices for which conditions (i) or (ii) do not hold. Set

$$v^{(i)} = s_i - s_{i-1} - 1 \ (i = 1 : u + 1), \quad s_0 = 0, \quad s_{n+1} = n + 1,$$

$$m^{(i)} = (m_{s_{i-1}+1}, ..., m_{s-1}), \quad \text{and}$$

$$x_{s_i} = \begin{cases} t_{k_{s_i}} & \text{if (i) does not hold} \\ t_{k_{s_i}+1} & \text{if (ii) does not hold.} \end{cases}$$

In accordance with parts 1 and 2 we construct $u + 1$ monosplines $M_i \in M^r_{v^{(i)}m^{(i)}}$ $(i = 1 : u + 1)$ such that

$$k(M_i, t_j) = \rho_j \qquad \text{for} \quad t_j \in (x_{s_{i-1}}, x_{s_i}),$$

$$k(M_i, x_{s_{i-1}}) = \begin{cases} r - m_{s_{i-1}} & \text{if } x_{s_{i-1}} = t_{k_{s_{i-1}}} \\ \rho_{k_{s_{i-1}}+1} - 1 & \text{if } x_{s_{i-1}} = t_{k_{s_{i-1}}+1}, \end{cases}$$

$$k(M_i, x_{s_i}) = \begin{cases} \rho_{k_{s_i}} - 1 & \text{if } x_{s_i} = t_{k_{s_i}} \\ r - m_{s_i} & \text{if } x_{s_i} = t_{k_{s_i}+1}. \end{cases}$$

The monospline $M(x) = M_i(x)$, $x \in (x_{s_{i-1}}, x_{s_i}]$, satisfies the requirements of the theorem. The theorem is proved.

Remark. The interlacing conditions hold if $\max_i \rho_i + \max_i m_i \leqslant r + 2$. Indeed, in this case

$$\rho_u \leqslant r + 2 - m_v \leqslant r + 1 \qquad (u = 1 : k; v = 1 : n). \tag{13}$$

Assume

$$k_i = \min \left(s : \sum_{j=1}^{s} \rho_j \geqslant \sum_{j=1}^{i} (m_j + 1) - 1 \right) \qquad (i = 1 : n). \tag{14}$$

If $k_i = 1$, then

$$\sum_{j=1}^{i} (m_j + 1) - 1 \leqslant \rho_1 \leqslant r + 1 \leqslant r - m_i + \sum_{j=1}^{i} (m_j + 1).$$

If $k_i > 1$, then

$$\sum_{j=1}^{k_i - 1} \rho_j \leqslant \sum_{j=1}^{i} (m_j + 1) - 2$$

and in view of (13)

$$\sum_{j=1}^{k_i} \rho_j \leqslant \rho_{k_i} + \sum_{j=1}^{i} (m_j + 1) - 2 \leqslant (r + 2 - m_i) + \sum_{j=1}^{i} (m_j + 1) - 2$$
$$= r - m_i + \sum_{j=1}^{i} (m_j + 1).$$

Hence, the multiplicities ρ_i, m_j, and indices (14) satisfy the interlacing conditions (2).

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